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A NOTE ON SPACES OF TEST AND GENERALIZED FUNCTIONS OF POISSON WHITE NOISE

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Abstract

The paper is devoted to construction and investigation of some riggings of the L^2-space of Poisson white noise. A particular attention is paid to the existence of a continuous version of a function from a test space, and to the property of an algebraic structure under pointwise multiplication of functions from a test space.

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1 Introduction

In the works [5, 6], started was a study of test and generalized functions defined on the Schwartz space of tempered distributions $\mathcal{S}'(\mathbb{R})$ the dual pairing of which is determined by the inner product of the L^2-space $(L^2_{\mathrm{P}}) = L^2(S'(\mathbb{R}), d\mu_{\mathrm{P}})$, where μ_{P} is the measure of Poisson white noise. Following the construction of the space of Hida distributions in Gaussian analysis (e.g., [10, 2, 4, 18]), Ito and Kubo [6] introduced the triple $(S_{\mathrm{P}})^* \supset (L^2_{\mathrm{P}}) \supset (S_{\mathrm{P}})$. However, the two following important problems remained open: 1) Does the space (S_{P}) consist of continuous functions, or, which is "almost" equivalent, do the delta functions belong to $(S_{\mathrm{P}})^*$? 2) Is the space (S_{P}) an algebra under pointwise multiplication of functions?

In this note, we will construct a whole scale of test spaces $(S_{\mathrm{P}})^{\varkappa}$, $\varkappa \geq 0$, such that $(S_{\mathrm{P}})^{\varkappa_1} \subset (S_{\mathrm{P}})^{\varkappa_2}$ if $\varkappa_1 > \varkappa_2$, and of their dual spaces $(S_{\mathrm{P}})^{-\varkappa}$ with respect to (L^2_{P}). For $\varkappa = 0$, $(S_{\mathrm{P}})^0 = (S_{\mathrm{P}})$, so that $(S_{\mathrm{P}})^{-0} = (S_{\mathrm{P}})^*$. The idea of construction of these spaces comes from the corresponding constructions in Gaussian analysis [8, 7].

The main results of the paper are the following: 1) The space $(S_{\mathrm{P}})^1$ consists of continuous functions, and for each $(S_{\mathrm{P}})^{\varkappa}$ with $\varkappa < 1$ this is not the case. 2) The space $(S_{\mathrm{P}})^1$ (and even each space $(S_{\mathrm{P}})^{\varkappa}$ with $\varkappa > 1$) is an algebra under pointwise multiplication, and, moreover, an estimate of Hilbert norms of a product of two functions from $(S_{\mathrm{P}})^1$ is given, this estimate being rather analogous to the estimates which hold for the Hida test space in Gaussian analysis, e.g., [11, 19, 20, 4].

It should be noted that there is another approach to developing analysis on non-Gaussian spaces, based on the use of a system of Appell polynomials and its dual (biorthogonal) system

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[1, 9]. By using Theorem 3.2 of the present paper, one can prove that the application of the biorthogonal analysis developed in [9] to the Poisson measure leads, in fact, to the same triple

$$(S_{\mathrm{P}})^{-1} \supset (L^2_{\mathrm{P}}) \supset (S_{\mathrm{P}})^{1}, \tag{1.1}$$

see [22, 15]. So, it seems that the triple (1.1) will play the role analogous to the Hida triple in Gaussian analysis, and one will use either the orthogonal or biorthogonal methods, or combine them, depending on a specific problem under study.

2 Setup for Poisson white noise calculus

In this section, we will construct the above mentioned scale of test and dual spaces. To this end, we present below some results of the works [5, 6, 14], see also [21].

Let T be a separable, topological space and ν a σ-finite, non-atomic measure defined on the Borel σ-algebra $\mathcal{B}(T)$. We consider a standard (Gelfand) triple of spaces [3]

$$\mathcal{E}' = \operatorname*{ind\,lim}_{p\to\infty} E_{-p} \supset L^2(T,\nu) = E_0 \supset \operatorname*{proj\,lim}_{p\to\infty} E_p = \mathcal{E}, \tag{2.1}$$

where $\{E_p \,|\, p \ge 0\}$ is a sequence of real, compatible, separable, Hilbert spaces such that, for any $p > q \ge 0$, E_p is topologically—that is, densely and continuously—embedded into E_q and

$$|\xi|_p \ge |\xi|_q, \qquad \xi \in E_p, \tag{2.2}$$

where $|\cdot|_p$ denotes the E_p norm; E_{-p} is the dual of E_p with respect to zero space E_0, so that $\mathcal{E}'$ is the dual of $\mathcal{E}$.

One makes the following assumptions about the space $\mathcal{E}$:

(A.1) Every element of E_1 has a version continuous on T, and for each $t \in T$, δ_t—the delta function at t—belongs to E_{-1}. Moreover, $\delta\colon t \to \delta_t$ is a continuous mapping of T into E_{-1} and

$$\|\delta\|^2 \equiv \int_T |\delta_t|^2_{-1}\, d\nu(t) < \infty,$$

so that [10] the embedding operator $E_1 \hookrightarrow E_0$ is of Hilbert–Schmidt type. Its Hilbert–Schmidt norm is less than 1.

(A.2) The mapping δ satisfies

$$\|\delta\|_\infty \equiv \int_T |\delta_t|_{-1}\, d\nu(t) + \sup_{t\in T} |\delta_t|_{-1} < \infty,$$

so that $E_1 \subset L^1(T,\nu) \cap L^\infty(T,\nu)$.

(A.3) For any $\xi \in \mathcal{E}$ and $p \ge 0$,

$$\rho|\xi|_{p+1} \ge |\xi|_p$$

with a fixed $\rho \in (0,1)$.

(A.4) The "diagonalization" operator $\mathfrak{D} : \mathcal{E}^{\otimes 2} \to \mathcal{E}$, $\mathcal{E}^{\otimes 2} = \operatorname*{proj\,lim}_{p\to\infty} E_p^{\otimes 2}$, given by

$$(\mathfrak{D} f^{(2)})(t) = f^{(2)}(t,t), \qquad f^{(2)} \in \mathcal{E}^{\otimes 2},\ t \in T, \tag{2.3}$$

is continuous, and moreover, for any $p \geq 1$,

$$|\mathfrak{D} f^{(2)}|_p \leq C_p |f^{(2)}\|_p, \qquad C_p > 0,$$

$|\cdot|_p$ standing also for the norm of each space $E_p^{\otimes n}$, so that $\mathcal{E}$ is an algebra under pointwise multiplication of functions and

$$|\xi\eta|_p \leq C_p |\xi|_p |\eta|_p, \qquad p \geq 1.$$

(A.5) The set of the functions $\xi \in \mathcal{E}$ whose support is of finite ν measure is dense in $\mathcal{E}$.

Let us construct an example of such a triple. Fix the sequence $(e_j)_{j=0}^{\infty}$ of the Hermite functions on $\mathbb{R}$:

$$e_j = e_j(t) = (\sqrt{\pi}\, 2^j j!)^{-1/2} (-1)^j e^{t^2/2} (d/dt)^j e^{-t^2}.$$

For each $p \geq 1$, define $\mathcal{S}_p(\mathbb{R})$ to be the real Hilbert space spanned by the orthonormal basis $\big(e_j (2j+2)^{-p}\big)_{j=0}^{\infty}$, and let $\mathcal{S}_p(\mathbb{R}^d) \equiv \mathcal{S}_p(\mathbb{R})^{\otimes d}$. Considered as a subset of $L^2(\mathbb{R}^d)$, every space $\mathcal{S}_p(\mathbb{R}^d)$ coincides with the domain of the operator $(H^{\otimes d})^p$, where $H^{\otimes d}$ is the harmonic oscillator in $L^2(\mathbb{R}^d)$: $H^{\otimes d} = -\sum_{i=1}^d \big(\frac{d}{dt_i}\big)^2 + \sum_{i=1}^d t_i^2 + 1$. As well known, $\mathcal{S}(\mathbb{R}^d) = \operatorname{proj\,lim}_{p\to\infty} \mathcal{S}_p(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. Denote by $\mathcal{S}_{-p}(\mathbb{R}^d)$ the dual of $\mathcal{S}_p(\mathbb{R}^d)$. Then, $\mathcal{S}_1(\mathbb{R}^d)$ consists of continuous functions and $\mathbb{R}^d \ni t \to \delta_t \in \mathcal{S}_{-1}(\mathbb{R}^d)$ is a continuous mapping such that $\sup_{t\in\mathbb{R}^d} \|\delta_t\|_{\mathcal{S}_{-1}(\mathbb{R}^d)} < \infty$. The assumptions (A.3), (A.4), and (A.5) are satisfied for $\mathcal{S}_p(\mathbb{R}^d)$'s. Let now ν be a σ-finite, non-atomic, Borel, regular, measure on $\mathbb{R}^d$. Suppose also that, for some $\varepsilon \geq 0$,

$$\int_{\mathbb{R}^d} \big(\|\delta_t\|^2_{\mathcal{S}_{-1-\varepsilon}(\mathbb{R}^d)} + \|\delta_t\|_{\mathcal{S}_{-1-\varepsilon}(\mathbb{R}^d)}\big)\, d\nu(t) < \infty \tag{2.4}$$

(for the Lebesgue measure, this holds when $\varepsilon = 0$). Making use of the evident estimate

$$\|\xi\|^2_{L^2(\mathbb{R}^d,\nu)} = \int_{\mathbb{R}^d} |\xi(t)|^2\, d\nu(t) \leq \|\xi\|_{\mathcal{S}_{1+\varepsilon}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \|\delta_t\|^2_{\mathcal{S}_{-1-\varepsilon}(\mathbb{R}^d)}\, d\nu(t),$$
$$\xi \in \mathcal{S}_{1+\varepsilon}(\mathbb{R}^d),$$

we conclude that $\mathcal{S}_{1+\varepsilon}(\mathbb{R}^d)$ is continuously embedded into $L^2(\mathbb{R}^d,\nu)$. Moreover, by (2.4) and [10], the embedding operator $O_{1+\varepsilon} : \mathcal{S}_{1+\varepsilon}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d,\nu)$ is of Hilbert–Schmidt type (note that, by passing to an equivalent system of norms, one can always make its Hilbert–Schmidt norm less than 1). Then because of the regularity and σ-finiteness of ν, $\mathcal{S}(\mathbb{R}^d)$ is a dense subset of $L^2(\mathbb{R}^d,\nu)$. At last, for each $p \geq 1$, define E_p to be the Hilbert factor space $\mathcal{S}_{p+\varepsilon}/\ker O_{p+\varepsilon}$, where $O_{p+\varepsilon}\colon \mathcal{S}_{p+\varepsilon}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d,\nu)$ is an embedding operator. By [2], Ch. 5, Sect. 5, subsec. 1, $\{L^2(T,\nu),\, E_p \,|\, p \geq 1\}$ is a sequence of compatible, Hilbert spaces, where T denotes the support of ν. Thus, we get the desired triple

$$\mathcal{S}'(T) = \mathcal{E}' = \operatorname*{ind\,lim}_{p\to\infty} E_{-p} \supset L^2(T,\nu) \supset \operatorname*{proj\,lim}_{p\to\infty} E_p = \mathcal{E} = \mathcal{S}(T). \tag{2.5}$$

Note that the spaces E_p, $p \geq 1$, are completely determined by the set T, and $\mathcal{E}$ is actually the test Schwartz space on T, $\mathcal{S}(T)$, respectively $\mathcal{E}'$ is the Schwartz space of tempered distributions on T, which is the dual of $\mathcal{S}(T)$ with respect to zero space $L^2(T,\nu)$. Note also that, in case of a bounded T, $\mathcal{S}(T) = \mathcal{D}(T)$ is the space of infinitely differentiable functions on T.

Given a real Hilbert space $\mathcal{H}$ and $\varkappa \in \mathbb{R}$, a weighted Fock space $\Gamma_\varkappa(\mathcal{H})$ is defined by

$$\Gamma_\varkappa(\mathcal{H}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} (n!)^{1+\varkappa},$$

where the symbol $\widehat{\otimes}$ denotes the symmetric tensor product, the index $\mathbb{C}$ stands for complexification of a real space, $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} 0} = \mathbb{C}$, $0! = 1$. Particularly, $\Gamma_0(\mathcal{H}) = \Gamma(\mathcal{H})$ is the usual Fock space over $\mathcal{H}$.

By using (2.1) and (2.2), we construct, for each $\varkappa \geq 0$, the following standard triple [10, 2, 4, 8, 7]

$$\Gamma_{-\varkappa}(\mathcal{E}') = \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{-\varkappa}(E_{-p}) \supset \Gamma(E_0) \supset \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\varkappa(E_p) = \Gamma_\varkappa(\mathcal{E}). \tag{2.6}$$

By (A.1), the embedding operator $\Gamma_\varkappa(E_1) \hookrightarrow \Gamma(E_0)$ is of Hilbert–Schmidt type, see, e.g., [2, 4].

We will use also the following triple

$$\Gamma_{\mathrm{fin}}(\mathcal{E})^* \supset \Gamma(E_0) \supset \Gamma_{\mathrm{fin}}(\mathcal{E}). \tag{2.7}$$

Here, $\Gamma_{\mathrm{fin}}(\mathcal{E})$ is the topological direct sum of the spaces

$$\mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n} \equiv \operatorname*{proj\,lim}_{p\to\infty} E_{p,\mathbb{C}}^{\widehat{\otimes} n}, \qquad n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\},$$

so that $\Gamma_{\mathrm{fin}}(\mathcal{E})$ consists of finite sequences $(f^{(n)})_{n=0}^{\infty}$, $f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n}$. The convergence in $\Gamma_{\mathrm{fin}}(\mathcal{E})$ is equivalent to the uniform finiteness and the coordinate-wise convergence in $\mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n}$. The $\Gamma_{\mathrm{fin}}(\mathcal{E})^*$ is the dual of $\Gamma_{\mathrm{fin}}(\mathcal{E})$ with respect to $\Gamma(E_0)$. It consists of all the sequences of the form $(F^{(n)})_{n=0}^{\infty}$, $F^{(n)} \in \mathcal{E}_{\mathbb{C}}'^{\widehat{\otimes} n}$, where

$$\mathcal{E}_{\mathbb{C}}'^{\widehat{\otimes} n} \equiv \operatorname*{ind\,lim}_{p\to\infty} E_{-p,\mathbb{C}}^{\widehat{\otimes} n}, \qquad n \in \mathbb{Z}_+.$$

The convergence in $\Gamma_{\mathrm{fin}}(\mathcal{E})^*$ is the coordinate-wise convergence in $\mathcal{E}_{\mathbb{C}}'^{\widehat{\otimes} n}$.

Now, endow $\mathcal{E}'$ with the strong dual topology, and define on the Borel σ-algebra $\mathcal{B}(\mathcal{E}')$ the probability measure μ_{P} by its Fourier transform

$$\int_{\mathcal{E}'} e^{i<x,\xi>} \, d\mu_{\mathrm{P}}(x) = \exp\left[\int_T \left(e^{i\xi(t)} - 1\right) d\nu(t)\right], \qquad \xi \in \mathcal{E},$$

μ_{P} is called the measure of Poisson white noise on T with intensity ν. Here, $< \cdot, \cdot >$ stands for dualization between the space $\mathcal{E}_{\mathbb{C}}'^{\widehat{\otimes} n}$ and $\mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n}$ for each n, which is supposed to be linear in both dots.

For any $x \in \mathcal{E}'$, the Poisson Wick power $:x^{\otimes n}: \in \mathcal{E}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, is defined by the recursion relation

$$\begin{gathered}
:x^{\otimes 0}: = 1, \qquad :x^{\otimes 1}: = x - 1, \\
< :x^{\otimes(n+1)}:, f^{(n+1)} > = < :x^{\otimes n}: \widehat{\otimes} :x^{\otimes 1}:, f^{(n+1)} > \\
-n < :x^{\otimes n}:, \mathfrak{D}^{(n+1)} f^{(n+1)} > -n < :x^{\otimes (n-1)}: \widehat{\otimes} \tau, f^{(n+1)} >, \\
f^{(n+1)} \in \mathcal{E}^{\widehat{\otimes}(n+1)}, \ n \in \mathbb{N},
\end{gathered} \tag{2.8}$$

where $\mathfrak{D}^{(n+1)}: \mathcal{E}^{\widehat{\otimes}(n+1)} \to \mathcal{E}^{\widehat{\otimes}n}$ is the continuous operator given by

$$\mathfrak{D}^{(2)} = \mathfrak{D},$$
$$\mathfrak{D}^{(n+1)} = \mathrm{id}^{\otimes(n-1)} \otimes \mathfrak{D} + \mathrm{id}^{\otimes(n-2)} \otimes \mathfrak{D} \otimes \mathrm{id} + \cdots + \mathfrak{D} \otimes \mathrm{id}^{\otimes(n-1)}, \qquad n \geq 2, \tag{2.9}$$

where id is the identity operator, $\mathfrak{D}$ is defined by (2.3), and τ is an element of $\mathcal{E}^{\widehat{\otimes}2}$ such that

$$< \tau, f^{(2)} >= \int_T \big(\mathfrak{D} f^{(2)}\big)(t)\, d\nu(t), \qquad f^{(2)} \in \mathcal{E}^{\widehat{\otimes}2}.$$

Remark 2.1. In order to distinguish between the Gaussian and Poisson Wick powers, it would be better to denote the latter by $:x^{\otimes n}:_{\mathrm{P}}$. But since the Gaussian Wick powers do not appear in this note, we do not use such a notation.

Evidently, for any $f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes}n}$, the dualization $< :x^{\otimes n}:, f^{(n)} >$ is well defined and is a continuous function of $x \in \mathcal{E}'$, which is called the Wick monomial with kernel $f^{(n)}$. Then, for each $f^{(n)} \in \widehat{L}^2(T^n, \nu^n) = \big(L^2(T,\nu)\big)_{\mathbb{C}}^{\widehat{\otimes}n}$, we define a function $< :x^{\otimes n}:, f^{(n)} >$ as an element of the space $(L^2_{\mathrm{P}}) = L^2(\mathcal{E}', d\mu_{\mathrm{P}})$ that is the (L^2_{P})-limit of an arbitrary sequence $\big(< :x^{\otimes n}:, f_j^{(n)} > \big)_{j=0}^{\infty}$ such that $f_j^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes}n}$ and $f_j^{(n)} \to f^{(n)}$ as $j \to \infty$ in $\widehat{L}^2(T^n, \nu^n)$.

Next, for any $f \in L^2(T,\nu) \cap L^1(T,\nu)$, we put

$$< x, f > \equiv < :x^{\otimes 1}:, f > + \int_T f(t)\, d\nu(t) \in (L^2_{\mathrm{P}}).$$

Hence, for any set $\alpha \subset T$ of finite ν measure, we can put $X_\alpha = X_\alpha(x) = < x, \chi_\alpha > \in (L^2_{\mathrm{P}})$, where χ_α is the indicator of α. Then, X_α is the Poisson random measure on T with intensity ν, i.e., for any $n \in \mathbb{N}$ and for arbitrary disjoint sets $\alpha_1, \dots, \alpha_n \in \mathcal{B}(T)$, the random variables $X_{\alpha_1}, \dots, X_{\alpha_n}$ are independent, and for each α X_α has the Poisson distribution with mean $\nu(\alpha)$. Thus, $\tilde{X}_\alpha = X_\alpha - \nu(\alpha) = < :x^{\otimes 1}:, \chi_\alpha >$ is the centered Poisson random measure.

Proposition 2.1 *For each* $f^{(n)} \in \widehat{L}^2(T^n, \nu^n)$,

$$< :x^{\otimes n}:, f^{(n)} >= \int_{T^n} f^{(n)}(t_1, \dots, t_n)\, d\tilde{X}_{t_1} \cdots d\tilde{X}_{t_n}, \tag{2.10}$$

where the right hand side of (2.10) *is the n-fold Wiener–Itô integral of* $f^{(n)}$ *by the centered Poisson random measure* $\tilde{X}_\alpha$.

Since a centered Poisson random measure has the chaotic representation property, Proposition 2.1 yields

Theorem 2.1 *The following mapping is a unitary*:

$$\Gamma(L^2(T,\nu)) \ni f = (f^{(n)})_{n=0}^{\infty} \to If = (If)(x) = \sum_{n=0}^{\infty} < :x^{\otimes n}:, f^{(n)} > \in (L^2_{\mathrm{P}}).$$

So, we are able now to construct different riggings of (L^2_{P}). We have only to apply the unitary I (or its extension by continuity) to the riggings of the Fock space $\Gamma(L^2(T,\nu))$ and get corresponding riggings of (L^2_{P}).

First, we note that

Proposition 2.2 *We have*

$$I\big(\Gamma_{\mathrm{fin}}(\mathcal{E})\big) = \mathcal{P}(\mathcal{E}'),$$

where $\mathcal{P}(\mathcal{E}')$ is the set of all continuous polynomials on $\mathcal{E}'$—that is, the set of all complex functions on $\mathcal{E}'$ of the form $\sum_{i=0}^{n} < x^{\otimes n}, f^{(n)} >$, $f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$.

Thus, the application of I to the rigging (2.7) gives

$$\mathcal{P}(\mathcal{E}')^* \supset (L^2_{\mathrm{P}}) \supset \mathcal{P}(\mathcal{E}').$$

Remark 2.2. The topology of the nuclear space $\mathcal{P}(\mathcal{E}')$ is supposed to be that induced from $\Gamma_{\mathrm{fin}}(\mathcal{E})$ by the isomorphism I. One can also define a nuclear topology on $\mathcal{P}(\mathcal{E}')$ from that of $\Gamma_{\mathrm{fin}}(\mathcal{E})$ by using the following natural isomorphism [2, 9]

$$\Gamma_{\mathrm{fin}} \ni f = (f^{(n)})_{n=0}^{\infty} \to \mathcal{U}f = (\mathcal{U}f)(x) = \sum_{n=0}^{\infty} < x^{\otimes n}, f^{(n)} > .$$

But, in fact, these two topologies coincide.

Thus, every generalized function Φ from the biggest (in a sense) space $\mathcal{P}(\mathcal{E}')^*$ can be represented in the form

$$\Phi = \Phi(x) = \sum_{n=0}^{\infty} < :x^{\otimes n}:, F^{(n)} >, \qquad F^{(n)} \in \mathcal{E}'^{\widehat{\otimes} n}_{\mathbb{C}},$$

and the dual pairing between Φ and a continuous polynomial

$$\phi(x) = \sum_{n=0}^{\infty} < :x^{\otimes n}:, f^{(n)} >, \qquad f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n},$$

is given by

$$\ll \Phi, \phi \gg = \sum_{n=0}^{\infty} < \overline{F^{(n)}}, f^{(n)} > n!,$$

$\overline{F^{(n)}}$ denoting the complex conjugate of $F^{(n)}$.

Next, by applying I to (2.6), we get

$$(\mathcal{E}_{\mathrm{P}})^{-\varkappa} = \operatorname*{ind\,lim}_{p\to\infty} (\mathcal{E}_{\mathrm{P}})^{-\varkappa}_{-p} \supset (L^2_{\mathrm{P}}) \supset \operatorname*{proj\,lim}_{p\to\infty} (\mathcal{E}_{\mathrm{P}})^{\varkappa}_{p} = (\mathcal{E}_{\mathrm{P}})^{\varkappa}, \qquad \varkappa \ge 0 \tag{2.11}$$

(we are using natural notations for all the images of the spaces from (2.6)). The norm of any Hilbert space $(\mathcal{E}_{\mathrm{P}})^{\sharp\varkappa}_{\sharp p}$, $p \ge 0$, $\varkappa \ge 0$, $\sharp \in \{+,-\}$, will be denoted by $\|\cdot\|_{\sharp\varkappa,\sharp p}$. The triple (2.11) with $\varkappa = 0$ was investigated in [6].

Let us consider two important examples of distributions. The first one is a Poisson white noise monomial:

$$:X'_{t_1}\cdots X'_{t_n} :=: x(t_1)\cdots x(t_n) :\equiv < :x^{\otimes n}:, \delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_n} >, \qquad t_1,\dots,t_n \in T. \tag{2.12}$$

By (A.1), the distribution (2.12) belongs to $(\mathcal{E}_{\mathrm{P}})^{-0}_{-1}$. The formulas (2.8) give the recursion relation:

$$\begin{aligned} :x(t_1)\cdots x(t_{n+1}): &= \big(:x(t_1)\cdots x(t_n)::x(t_{n+1}):\big)\widehat{\ } - n\big(:x(t_1)\cdots x(t_n): \delta(t_n - t_{n+1})\big)\widehat{\ } \\ &\quad - n\big(:x(t_1)\cdots x(t_{n-1}): 1(t_n)\delta(t_n - t_{n+1})\big)\widehat{\ }, \end{aligned}$$

where $\delta = \delta_0$ is the delta function at 0, the index $\hat{}$ stands for the symmetrization of a function.

The second example is the Poisson white noise exponential function

$$:e^{<x,y>} :\equiv \sum_{n=0}^{\infty}(n!)^{-1} < :x^{\otimes n}:, y^{\otimes n} >, \qquad y \in \mathcal{E}'_{\mathbb{C}}. \tag{2.13}$$

If $y \in E_{-p,\mathbb{C}}$, $p > 0$, then $:e^{<x,y>}:\in (\mathcal{E}_{\mathrm{P}})^{-0}_{-p}$; if $\xi \in E_{p,\mathbb{C}}$, $p \geq 0$, then $:e^{<x,\xi>}:\in (\mathcal{E}_{\mathrm{P}})^{\varkappa}_{p}$ with $\varkappa < 1$, and

$$:e^{<x,\xi>}:\in (\mathcal{E}_{\mathrm{P}})^{1}_{p} \quad \text{if } |\xi|_p < 1 \tag{2.14}$$

(we keep the notation $|\cdot|_p$ for the complexified spaces $E^{\hat{\otimes} n}_{p,\mathbb{C}}$). We will return to this function in the next section.

3 Continuous version theorem and Poisson white noise delta function

In this section, we will show that every element of the space $(\mathcal{E}_{\mathrm{P}})^1$ has a version (in the (L^2_{P})-sense) whose restriction to every E_{-p} is continuous on E_{-p}. Such a continuity will be called a continuity on $\mathcal{E}'$ (though it does not imply the continuity on $\mathcal{E}'$ endowed with the strong dual topology). This will allow us to introduce a Poisson white noise delta function. Also, by using the (proof of the) theorem on continuity, we will obtain a theorem on the explicit form of the Poisson white noise exponential function.

Theorem 3.1 *Each ϕ from the space $(\mathcal{E}_{\mathrm{P}})^1$ has a version $\tilde{\phi}$ that is continuous on $\mathcal{E}'$ and is given by*

$$\tilde{\phi}(x) = \sum_{n=0}^{\infty} < :x^{\otimes n}:, f^{(n)} >, \qquad f^{(n)} \in \mathcal{E}^{\hat{\otimes} n}_{\mathbb{C}}, \tag{3.1}$$

where the Wick powers $:x^{\otimes n}:$ are defined by the recursion relation (2.8) *and ϕ is the image under the unitary I of the element $(f^{(n)})_{n=0}^{\infty}$ of the space $\Gamma_1(\mathcal{E})$. The series* (3.1) *converges absolutely and uniformly on every bounded set from $\mathcal{E}'$. Moreover, $\tilde{\phi}(x)$ can be extended to the complexification $\mathcal{E}'_{\mathbb{C}}$ of $\mathcal{E}'$ that $\tilde{\phi}(z)$ becomes analytic in $\mathcal{E}'_{\mathbb{C}}$. This extension is given by the formulas* (3.1) *and* (2.8) *in which $x \in \mathcal{E}'$ is replaced by $z \in \mathcal{E}'_{\mathbb{C}}$.*

Remark 3.1. The definition of a function analytic in $\mathcal{E}'_{\mathbb{C}}$ can be found, e.g., in [13, 4].

Proof. Our proof is close in spirit to (part of) the proof of the continuous version theorem in Gaussian analysis proposed by Obata [17, 18]. We wish to estimate the norms of $:x^{\otimes n}:$ in $E^{\hat{\otimes} n}_{-p}$.

Since $< \tau, \xi^{\otimes 2} > = < \int_T \delta_t^{\otimes 2}\, d\nu(t), \xi^{\otimes 2} >$, we get from (A.1) and (2.2) that

$$|\tau|_{-p} \leq \int_T |\delta_t|^2_{-p}\, d\nu(t) \leq \int_T |\delta_t|^2_{-1}\, d\nu(t) = \|\delta\|^2, \qquad p \geq 1. \tag{3.2}$$

By (A.4) and (2.9), we easily conclude that

$$|\mathfrak{D}^{(n+1)} f^{(n+1)}|_p \leq nC_p |f^{(n+1)}|_p, \qquad p \geq 1, \tag{3.3}$$

i.e., $\mathfrak{D}^{(n+1)}\colon E^{\hat{\otimes}(n+1)}_p \to E^{\hat{\otimes} n}_p$ is a linear continuous operator with norm $\leq nC_p$.

By (2.8),

$$:x^{\otimes n+1}: = :x^{\otimes n}:\widehat{\otimes}:x^{\otimes 1}: - \mathfrak{D}^{(n+1)*}:x^{\otimes n}: - n:x^{\otimes(n-1)}:\widehat{\otimes}\tau, \tag{3.4}$$

where $\mathfrak{D}^{(n+1)*}\colon \mathcal{E}'^{\widehat{\otimes}n} \to \mathcal{E}^{\widehat{\otimes}(n+1)}$ is the dual operator of $\mathfrak{D}^{(n+1)}$, and for each $p \geq 1$, $\mathfrak{D}^{(n+1)}\colon E_{-p}^{\widehat{\otimes}n} \to E_{-p}^{\widehat{\otimes}(n+1)}$ is a continuous operator with norm $\leq nC_p$.

By (A.2) and (2.2), we have that $1 \in E_{-1}$ and

$$|1|_{-p} \leq |1|_{-1} \leq \int_T |\delta_t|_{-1}\, d\nu(t) \leq \|\delta\|_\infty, \qquad p \geq 1. \tag{3.5}$$

Thus, by (3.2)–(3.5),

$$\begin{aligned}|:x^{\otimes(n+1)}:|_{-p} &\leq |:x^{\otimes n}:|_{-p}(|x|_{-p} + \|\delta\|_\infty) \\ &\quad + nC_p|:x^{\otimes n}:|_{-p} + \|\delta\|^2|:x^{\otimes(n+1)}:|_{-p}, \qquad p \geq 1.\end{aligned} \tag{3.6}$$

Hence, given any fixed $p \geq 1$ and $R > 0$, we have, for each $x \in \mathcal{E}'$ such that $|x|_{-p} \leq R$,

$$\begin{gathered}|:x^{\otimes(n+1)}:|_{-p} \leq nY_{p,R}\max\{|:x^{\otimes n}:|_{-p}, |:x^{\otimes(n-1)}:|_{-p}\}, \\ Y_{p,R} = R + \|\delta\|_\infty + C_p + \|\delta\|^2,\end{gathered}$$

which easily yields that

$$|:x^{\otimes(n+1)}:|_{-p} \leq n!\, Z_{p,R}^n, \qquad Z_{p,R} = \max\{1, Y_{p,R}\}. \tag{3.7}$$

The assumption (A.3) implies

$$\rho^{mn}|f^{(n)}|_{p+m} \geq |f^{(n)}|_p, \qquad f^{(n)} \in E_p^{\widehat{\otimes}n},\ m \in \mathbb{N}, \tag{3.8}$$

and so

$$|F^{(n)}|_{-(p+m)} \leq \rho^{mn}|F^{(n)}|_{-p}, \qquad F^{(n)} \in E_{-p}^{\widehat{\otimes}n},\ m \in \mathbb{N}. \tag{3.9}$$

Summing (3.7) and (3.9) up, we conclude that there exists $p_1 = p_1(p,R) \geq p$ such that

$$|:x^{\otimes n}:|_{-p_1} \leq n!\, 2^{-n}.$$

Therefore, for $\phi(x) = \sum_{n=0}^\infty < :x^{\otimes n}:, f^{(n)} > \in (\mathcal{E}_{\mathrm{P}})^1$, we have, for $|x|_{-p} \leq R$,

$$\begin{aligned}\sum_{n=0}^\infty |< :x^{\otimes n}:, f^{(n)} >| &\leq \sum_{n=0}^\infty |:x^{\otimes n}:|_{-p_1}|f^{(n)}|_{p_1} \leq \sum_{n=0}^\infty n!\, 2^{-n}|f^{(n)}|_{p_1} \\ &\leq \left(\sum_{n=0}^\infty 4^{-n}\right)^{1/2}\|\phi\|_{-1,-p_1},\end{aligned}$$

i.e., the series $\sum_{n=0}^\infty < :x^{\otimes n}:, f^{(n)} >$ converges absolutely and uniformly on every bounded set in E_{-p} (we recall that every bounded set in $\mathcal{E}'$ endowed with the strong dual topology is bounded in some space E_{-p} [3]). For any $f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes}n}$, the function $< :x^{\otimes n}:, f^{(n)} >$ is a continuous polynomial of variable $x \in \mathcal{E}'$ (see Proposition 2.2), and so the function $\tilde{\phi}(x)$ is continuous on every E_{-p}.

Next, we note that if one replaces $x \in \mathcal{E}'$ with $z \in \mathcal{E}'_{\mathbb{C}}$ in (2.8) and (3.1), then all the above formulas hold true for the complexified spaces $E_{-p,\mathbb{C}}$. Since $<: z^{\otimes n} :, f^{(n)} >$, $f^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\widehat{\otimes} n}$, is a continuous polynomial of variable z, it is an analytic function in every $E_{-p,\mathbb{C}}$. The series

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} <: z^{\otimes n} :, f^{(n)} >$$

converges absolutely and uniformly on every bounded set in $E_{-p,\mathbb{C}}$, so $\tilde{\phi}(z)$ is analytic in every $E_{-p,\mathbb{C}}$. By [13], Theorem A.2, we conclude that $\tilde{\phi}(z)$ is analytic in $\mathcal{E}'_{\mathbb{C}}$. □

Having obtained the continuous version theorem, we are able now to define a Poisson white noise delta function. So, we put, for each $y \in \mathcal{E}'$,

$$\tilde{\delta}_y = \sum_{n=0}^{\infty} <: x^{\otimes n} :, (n!)^{-1} : y^{\otimes n} :> .$$

Proposition 3.1 *For each $y \in \mathcal{E}'$, $\tilde{\delta}_y$ belongs to $(\mathcal{E}_{\mathrm{P}})^{-1}$, and for each $\phi \in (\mathcal{E}_{\mathrm{P}})^{1}$,*

$$\ll \tilde{\delta}_y, \phi \gg = \tilde{\phi}(y), \tag{3.10}$$

where $\tilde{\phi}$ is the continuous version of ϕ defined in Theorem 3.1. *Moreover, the following mapping is continuous*:

$$\mathcal{E}' \ni y \to \tilde{\delta}_y \in (\mathcal{E}_{\mathrm{P}})^{-1}.$$

Proof. Let us fix $y \in \mathcal{E}'$, and let $p > 0$ be such that $y \in E_{-p}$, $|y|_{-p} = R$. Let $p_1 \ge p$ be chosen so that

$$(n!)^{-1} | : y^{\otimes n} : |_{-p_1} \le 2^{-n} \tag{3.11}$$

(see the proof of Theorem 3.1). Then

$$\|\tilde{\delta}_y\|^2_{-1,-p_1} = \sum_{n=0}^{\infty} (n!)^{-2} | : y^{\otimes n} : |^2_{-p_1} \le \sum_{n=0}^{\infty} 4^{-n} < \infty,$$

whence $\tilde{\delta}_y \in (\mathcal{E}_{\mathrm{P}})^{-1}_{-p_1}$ and (3.10) holds.

Let a sequence $\{y_j \,|\, j \in \mathbb{N}\} \in E_{-p}$ tend to a y in E_{-p}. Then, there is $R > 0$ such that $|y_j|_{-p} \le R$ for all $j \in \mathbb{N}$, and so $|y|_{-p} \le R$. Choose p_1 so that (3.11) holds for any $y \in E_{-p}$ with $|y|_{-p} \le R$. Let us show that

$$\tilde{\delta}_{y_j} \to \tilde{\delta}_y \quad \text{in } (\mathcal{E}_{\mathrm{P}})^{-1}_{-p_1} \text{ as } j \to \infty. \tag{3.12}$$

As follows from the proof of Theorem 3.1,

$$: y_j^{\otimes n} : \to : y^{\otimes n} : \quad \text{in } E_{-p_1}^{\widehat{\otimes} n},\ n \in \mathbb{Z}_+,$$

whence

$$<: x^{\otimes n} :, (n!)^{-1} : y_j^{\otimes n} :> \to <: x^{\otimes n} :, (n!)^{-1} : y^{\otimes n} :> \quad \text{in } (\mathcal{E}_{\mathrm{P}})^{-1}_{-p_1} \text{ as } j \to \infty,\ n \in \mathbb{Z}_+. \tag{3.13}$$

Next, for any $n,m \in \mathbb{N}$, $n > m$, we get by (3.11)

$$\begin{aligned}\sum_{i=m}^{n} \| <:x^{\otimes i}:, (i!)^{-1}(:y_j^{\otimes i}: - :y^{\otimes i}:) > \|_{-1,-p_1}^2 &\le \sum_{i=m}^{n} 2(i!)^{-2}\big(|\, :y_j^{\otimes i}: |_{-p_1}^2 + |\, :y^{\otimes i}: |_{-p_1}^4\big) \\ &\le 4\sum_{i=m}^{n} 4^{-i}.\end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we easily conclude that (3.12) holds. $\square$

Now, we will prove a theorem on the evident form of the Poisson white noise exponential function. This theorem is a refinement of a corresponding result of [6].

Theorem 3.2 *We have*

$$\begin{gathered} :e^{<x,\xi>} := \exp\Big[< x, \log(1+\xi) > - \int_T \xi(t)\, d\nu(t)\Big], \\ \xi \in \mathcal{E}_{\mathbb{C}},\ \ |\xi|_1 < \max\{1, C_1\}, \end{gathered} \tag{3.15}$$

which holds for μ_{P}-a.a. $x \in \mathcal{E}'$, more exactly, for all $x \in E_{-1}$, which is a set of full μ_{P} measure.

Proof. We divide the proof into steps.

1. Let us fix an arbitrary set $\alpha \subset T$ of finite ν measure and put

$$f(t) = \sum_{j=1}^{k} \omega_j \chi_{\alpha_j}(t), \qquad \omega_j \in \mathbb{C},$$

where α_j, $j = 1, \dots, k$, are disjoint subsets of α, $\bigcup_{j=1}^k \alpha_j = \alpha$. Then, by Proposition 2.1,

$$\begin{gathered} <:x^{\otimes n}:, \chi_{\alpha_1}^{\otimes n_1} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{\alpha_k}^{\otimes n_k} > = \prod_{j=1}^{k} <:x^{\otimes n_j}:, \chi_{\alpha_j}^{\otimes n_j} >, \quad n_1 + \cdots + n_k = n, \\ <:x^{\otimes n_j}:, \chi_{\alpha_j}^{\otimes n_j} > = C_{n_j}(< x, \chi_{\alpha_j} >, \nu(\alpha_j)), \end{gathered} \tag{3.16}$$

where $C_n(u,\tau)$ are the Charlier polynomials with parameter τ:

$$\sum_{n=0}^{\infty} \frac{\omega^n}{n!} C_n(u,\tau) = \exp\big[u \log(1+\omega) - \omega\tau\big]. \tag{3.17}$$

Therefore, by (3.16), (3.17), we have (cf. [5, 6])

$$\begin{aligned} &\sum_{n=0}^{\infty} (n!)^{-1} <:x^{\otimes n}:, f^{\otimes n} > \\ &\quad = \exp\Big[< x, \sum_{j=1}^{k} \chi_{\alpha_j} \log(1+\omega_j) > - \int_T \Big(\sum_{j=1}^{k} \omega_j \chi_{\alpha_j}(t)\Big) d\nu(t)\Big] \\ &\quad = \exp\Big[< x, \chi_\alpha \log(1+f) > - \int_T f(t)\, d\nu(t)\Big], \end{aligned} \tag{3.18}$$

the equalities making sense for $|\omega_j| < 1$.

2. For any $n \in \mathbb{N}$, let us consider the following triple

$$\big(\widehat{L}^\infty(\alpha^n, \nu^n)\big)' \supset \widehat{L}^2(\alpha^n, \nu^n) \supset \widehat{L}^\infty(\alpha^n, \nu^n),$$

where $\widehat{L}^\infty(\alpha^n, \nu^n)$ is the subspace of $L^\infty(\alpha^n, \nu^n)$ consisting of symmetric functions on α^n, and $\big(\widehat{L}^\infty(\alpha^n, \nu^n)\big)'$ is the dual of $\widehat{L}^\infty(T^n, \nu^n)$ with respect to zero space $\widehat{L}^2(\alpha^n, \nu^n)$. Let us fix an arbitrary $x = x(t) \in L^1(\alpha, \nu)$. We suppose that $:x^{\otimes n}:$ are defined by the recursion relation (2.8) and these $:x^{\otimes n}:$ are understood as elements of $\big(\widehat{L}^\infty(\alpha^n, \nu^n)\big)'$. By analogy with the proof of Theorem 3.1, the following estimate can be proved

$$\| :x^{\otimes n}: \|_{\left(\widehat{L}^\infty(\alpha^n, \nu^n)\right)'} \le n!\, V^n,$$

where V is a positive constant, depending on α and x. Therefore, for any $f \in L^\infty(\alpha, \nu)$, we have

$$\sum_{n=0}^{\infty} (n!)^{-1} | <:x^{\otimes n}:, f^{\otimes n}> | \le \sum_{n=0}^{\infty} \big(V\|f\|_{L^\infty(\alpha,\nu)}\big)^n.$$

Hence, $x \in L^1(\alpha, \nu)$ being fixed,

$$:e^{<x,f>}:= \sum_{n=0}^{\infty} (n!)^{-1} <:x^{\otimes n}:, f^{\otimes n}>$$

is a continuous function of variable $f \in L^\infty(\alpha, \nu)$ in the ball in $L^\infty(\alpha, \nu)$ of radius $(2V)^{-1}$ centered at 0.

Let us fix $\xi \in \mathcal{E}_{\mathbb{C}}$ such that $|\xi|_1 \le (2V\|\delta\|_\infty)^{-1}$, and so, by (A.2), $\|\xi\|_{L^\infty(T,\nu)} \le (2V)^{-1}$. Approximate the function $\xi\chi_\alpha$ in the $L^\infty(T, \nu)$ norm by step functions. Since (3.18) holds for all $x \in L^1(T, \nu)$, we conclude that (3.18) holds true for our fixed x if f is replaced by $\xi\chi_\alpha$.

3. Let us denote by $f(x; \xi)$ the function on the right hand side of (3.15). By virtue of (A.4), $\log(1 + \xi) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \xi^{\otimes n} \in E_{1,\mathbb{C}}$ provided $\xi \in \mathcal{E}$ and $|\xi|_1 < R = \max\{1, C_1\}$. Thus, taking also to notice (A.2), we conclude that $f(x; \xi)$ is well defined for all $x \in E_{-1}$ and the above ξ. Moreover, for each $x \in E_{-1}$ fixed, $f(x; \xi)$ as a function of $\xi \in E_{1,\mathbb{C}}$ is analytic in the ball $|\xi|_1 < R$. Therefore, see, e.g., the proof of Theorem 3 in [7], it can be represented in the form

$$f(x; \xi) = \sum_{n=0}^{\infty} (n!)^{-1} < F^{(n)}(x), \xi^{\otimes n} >, \qquad F^{(n)}(x) \in \mathcal{E}'^{\widehat{\otimes} n}_{\mathbb{C}}.$$

Moreover, since $f(x; \xi) \in \mathbb{R}$ for any $\xi \in \mathcal{E}$, we conclude that actually $F^{(n)}(x) \in \mathcal{E}'^{\widehat{\otimes} n}$, i.e., $< F^{(n)}(x), f^{(n)} > \in \mathbb{R}$ for any $f^{(n)} \in \mathcal{E}^{\widehat{\otimes} n}$.

As follows from 1–2, for any $x \in L^1(T, \nu)$ and any set α of finite ν measure, we have

$$:e^{<x,\xi>}:= \sum_{n=0}^{\infty} (n!)^{-1} <:x^{\otimes n}:, \xi^{\otimes n}>= \sum_{n=0}^{\infty} (n!)^{-1} < F^{(n)}(x), \xi^{\otimes n} >= f(x; \xi),$$

where $\xi \in \mathcal{E}$ is such that it vanishes outside of α and $|\xi|_1 \le \mathcal{Y}_\alpha$, $\mathcal{Y}_\alpha$ a positive constant determined by x and α. From here, because of (A.5), $:x^{\otimes n}:= F^{(n)}(x)$. Thus, (3.15) holds for all $x \in L^1(T, \nu)$.

4. It follows from the proof of Theorem 3.1 that there exist $p_0 \geq 1$ and $r > 0$ such that, for any fixed $\xi \in \mathcal{E}$, $|\xi|_{p_0} \leq r$, $:e^{<x,\xi>}:$ is a continuous function of variable $x \in E_{-1}$. On the other hand, $f(x;\xi)$ is continuous on E_{-1} for all $\xi \in \mathcal{E}$, $|\xi|_1 < R$. Let us fix an arbitrary $\xi \in \mathcal{E}$, $|\xi|_{p_0} \leq r$. We know that

$$:e^{<x,\xi>}:= f(x;\xi) \quad \text{for all } x \in L^1(T,\nu) \tag{3.19}$$

(we assume that $r \leq R$). Extending (3.19) by continuity (since $\mathcal{E} \subset L^1(T,\nu)$ and $\mathcal{E}$ is a dense subset of E_{-1}, $L^1(T,\nu)$ is also a dense subset of E_{-1}), we conclude that (3.19) holds true for all $x \in E_{-1}$ and $\xi \in \mathcal{E}$, $|\xi|_{p_0} \leq r$. At last, analogously to 3, we get the conclusion of the theorem. $\square$

Remark 3.2. It follows from the proof of Theorem 3.2 that the formula (3.15) holds for all $x \in E_{-p}$, $p \geq 1$, provided $\xi \in \mathcal{E}_{\mathbb{C}}$ and $|\xi|_p < \big(\max\{1, C_p\}\big)^{-1}$.

Corollary 3.1 *For each $y \in \mathcal{E}'$, $y \neq 0$, the Poisson white noise delta function $\tilde{\delta}_y$ does not belong to any space $(\mathcal{E}_{\mathrm{P}})^{\varkappa}$ with $\varkappa < 1$.*

Remark 3.3. This statement shows that there is no sense in trying to prove the continuous version theorem for any $(\mathcal{E}_{\mathrm{P}})^{-\varkappa}$ with $\varkappa < 1$.

Proof. Following [6, 8], for any $\Phi \in (\mathcal{E}_{\mathrm{P}})^{-\varkappa}$ with $\varkappa < 1$, we define the $\mathcal{S}_{\mathrm{P}}$-transform of Φ by

$$\mathcal{S}_{\mathrm{P}}[\Phi](\xi) = \langle\!\langle \Phi, :e^{<x,\xi>}: \rangle\!\rangle, \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

Since the set of $:e^{<x,\xi>}:$ is total in each $(\mathcal{E}_{\mathrm{P}})^{-\varkappa}$, the $\mathcal{S}_{\mathrm{P}}$-transform uniquely defines Φ. Moreover, for each $\Phi \in (\mathcal{E}_{\mathrm{P}})^{-1}$, there is $p > 0$ such that $\Phi \in (\mathcal{E}_{\mathrm{P}})^{-1}_{-p}$, and we set, taking to notice (2.14),

$$\mathcal{S}_{\mathrm{P}}[\Phi](\xi) = \langle\!\langle \Phi, :e^{<x,\xi>}: \rangle\!\rangle, \qquad \xi \in \mathcal{E}_{\mathbb{C}},\ |\xi|_p < 1. \tag{3.20}$$

As follows from [7], the $\mathcal{S}_{\mathrm{P}}$-transform defined by (3.20) uniquely determines $\Phi \in (\mathcal{E}_{\mathrm{P}})^{-1}$.

Since all the above constructions are totally isomorphic to the Gaussian case, all the characterization theorems of the spaces $(\mathcal{E}_{\mathrm{P}})^{\varkappa}$, $\varkappa \in [-1,1]$, in terms of their $\mathcal{S}_{\mathrm{p}}$-transforms hold true, e.g., [2, 4, 18, 8, 7].

Let us fix $y \in \mathcal{E}'$, $y \neq 0$, and let $p > 0$ be such that $y \in E_{-p}$. By virtue of Proposition 3.1, we know that *a priori* $\tilde{\delta}_y \in (\mathcal{E}_{\mathrm{P}})^{-1}$. By Theorem 3.2 (more exactly, by Remark 3.2), we get that $\mathcal{S}_{\mathrm{P}}[\tilde{\delta}_y](\xi)$ is equal to the right hand side of (3.15) provided $|\xi|_p < \max\{1, C_p\}^{-1}$. But the function $\mathcal{S}_{\mathrm{P}}[\tilde{\delta}_y](\xi)$ is analytic only in a neighborhood of zero in $\mathcal{E}_{\mathbb{C}}$, but not in the whole $\mathcal{E}_{\mathbb{C}}$. Therefore, by the characterization theorems, which state particularly that the $\mathcal{S}_{\mathrm{P}}$-transform of an element of $(\mathcal{E}_{\mathrm{P}})^{-\varkappa}$ with $\varkappa < 1$ is a function analytic in $\mathcal{E}_{\mathbb{C}}$, we obtain the desired statement. $\square$

Corollary 3.2 *For each $t \in T$, define a linear continuous operator ∂_t in $(\mathcal{E}_{\mathrm{P}})^1$ by*

$$\partial_t <:x^{\otimes n}:, f^{(n)}> = n <:x^{\otimes (n-1)}:, f^{(n)}(\cdot, \dots, \cdot, t)>,$$

∂_t is called the operator of Hida differentiation at t. Then, for each $\phi \in (\mathcal{E}_{\mathrm{P}})^1$,

$$\big(\partial_t \phi\big)^{\sim}(x) = \tilde{\phi}(x + \delta_t) - \tilde{\phi}(x), \qquad x \in \mathcal{E}',$$

where $\tilde{\phi}$ denotes the continuous version of ϕ defined in Theorem 3.1.

Proof. For $\phi \in \mathcal{P}(\mathcal{E}')$, the corollary was proved in [5, 6]. For an arbitrary $\phi \in (\mathcal{E}_{\mathrm{P}})^1$, let us choose a sequence $(\phi_j)_{j=1}^{\infty} \in \mathcal{P}(\mathcal{E}')$ such that $\phi_j \to \phi$ in $(\mathcal{E}_{\mathrm{P}})^1$. Then, by Proposition 3.1,

$$\tilde{\phi}_j(x+\delta_t) - \tilde{\phi}_j(x) = \ll \tilde{\delta}_{x+\delta_t} - \tilde{\delta}_x, \phi_j \gg \to \ll \tilde{\delta}_{x+\delta_t} - \tilde{\delta}_x, \phi \gg = \tilde{\phi}(x+\delta_t) - \tilde{\phi}(x). \tag{3.21}$$

On the other hand,

$$\tilde{\phi}_j(x+\delta_t) - \tilde{\phi}_j(x) = (\partial_t \phi_j)^{\sim}(x) = \ll \tilde{\delta}_x, \partial_t \phi_j \gg \to \ll \tilde{\delta}_x, \partial_t \phi \gg = (\partial_t \phi)^{\sim}(x). \tag{3.22}$$

Combining (3.21) and (3.22) gives the corolary. □

Remark 3.4. It is worth to compare Corollary 3.2 with the results of Nualart and Vives [16]

4 Multiplication in $(\mathcal{E}_{\mathrm{P}})^1$

Theorem 4.1 *The space $(\mathcal{E}_{\mathrm{P}})^1$ is an algebra under pointwise multiplication of functions. More exactly, for any $\phi, \psi \in (\mathcal{E}_{\mathrm{P}})^1$ and $p \geq 1$, there is* const > 0 *such that*

$$\|\phi\psi\|_{1,p} \leq \text{const}\, \|\phi\|_{1,p+1} \|\psi\|_{1.p+q},$$

where $q \in \mathbb{N}$ is chosen so that

$$\rho^q < (1-\rho)^2 Y_p^{-1}, \qquad Y_p = \max\{1, C_p\} \max\{1, \|\delta\|_\infty\},$$

ρ the constant from (A.3). *In particular, for any $\phi \in (\mathcal{E}_{\mathrm{P}})^1$, the operator of multiplication by ϕ acts continuously from $(\mathcal{E}_{\mathrm{P}})^1_{p+1}$ into $(\mathcal{E}_{\mathrm{P}})^1_p$ for each $p \geq 1$.*

Proof. The proof is rather analogous to that of Proposition 6.5 in [6], so we only note some new points.

Let $f^{(n)} \in \mathcal{E}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Then, for the operator $A_{j^*,k,j}(f^{(n)})$, $j^*+k+j=n$, defined in [6], we have the estimate (compare with Proposition 4.5 in [6]):

$$\|A_{j^*,k,j}(f^{(n)})\phi\|_{1,p} \leq \rho^{(j^*+k)}(j^*+k)!\,(1-\rho)^{-(j^*+k+1)} C_p^{k+j} \|\delta\|_\infty^j \rho^{j(p-1)} |f^{(n)}|_p \|\phi\|_{1,p+1}, \tag{4.1}$$

so that $A_{j^*,k,j}(f^{(n)})$ is a continuous operator in $(\mathcal{E}_{\mathrm{P}})^1$.

Let $\psi \in (\mathcal{E}_{\mathrm{P}})^1$ be of the form $\psi(x) = \sum_{n=0}^{\infty} <:x^{\otimes n}:, f^{(n)}>$. Then [6]

$$<:x^{\otimes n}:, f^{(n)}> \phi(x) = \sum_{j^*+k+j=n} \frac{n!}{j^*!\,k!\,j!} A_{j^*,k,j}(f^{(n)})\phi. \tag{4.2}$$

By (4.1) and (4.2)

$$\|\phi\psi\|_{1,p} \leq \|\phi\|_{1,p+1} \sum_{n=0}^{\infty} n!\,(1-\rho)^{-(n-1)} Y_p^n |f^{(n)}|_p \sum_{j^*+k+j=n} \frac{(j^*+k)!}{j^*!\,k!\,j!}\,\rho^k. \tag{4.3}$$

By using the estimate (3.25) in [6], we have

$$\frac{1}{j^*!\,j!} \sup_{k\geq 0} \frac{(j^*+k)!}{k!}\,\rho^k \leq \frac{1}{j!}\,(1-\rho)^{-j^*-1},$$

which, upon (4.3) and (3.8), gives

$$\|\phi\psi\|_{1,p} \le \|\phi\|_{1,p+1} \sum_{n=0}^{\infty} n!\,(1-\rho)^{-2(n+1)} Y_p^n \rho^{nq} |f^{(n)}|_{p+q}\, \tfrac{1}{2}(n+1)(n+2)$$
$$\le \tfrac{1}{2}\|\phi\|_{1,p+1}\|\psi\|_{1,p+q}(1-\rho)^{-2}\biggl(\sum_{n=0}^{\infty}(n+1)^2(n+2)^2\bigl[(1-\rho)^{-2}Y_p\rho^q\bigr]^{2n}\biggr)^{1/2},$$

which gives the theorem. □

Remark 4.1. By analogy with the proof of Theorem 4.1, one can easily verify that each space $(\mathcal{E}_{\mathrm{P}})^{\varkappa}$ with $\varkappa > 1$ has an algebraic structure.

Following [6], for each $t \in T$, we define the operator of Poisson coordinate multiplication

$$x(t)\cdot = (\partial_t^* + 1)(\partial_t - 1) = \partial_t^*\partial_t + \partial_t + \partial_t^* + 1,$$

where $\partial_t^* \colon (\mathcal{E}_{\mathrm{P}})^{-1} \to (\mathcal{E}_{\mathrm{P}})^{-1}$ is the dual of the operator $\partial_t \colon (\mathcal{E}_{\mathrm{P}})^{1} \to (\mathcal{E}_{\mathrm{P}})^{1}$ defined in Corollary 3.2. Evidently, $x(t)\cdot$ is a continuous operator from $(\mathcal{E}_{\mathrm{P}})^{1}$ to $(\mathcal{E}_{\mathrm{P}})^{-1}$.

Corollary 4.1 *For $\phi, \psi \in (\mathcal{E}_{\mathrm{P}})^{1}$ and $t \in T$,*

$$\ll x(t)\cdot\phi, \psi \gg = \ll x(t), \overline{\phi}\psi \gg,$$

where $\overline{\phi}$ is the complex conjugate of ϕ. Here, $x(t) = <:x^{\otimes 1}:, \delta_t> + 1 \in (\mathcal{E}_{\mathrm{P}})^{-1}$.

Proof. Let us choose an arbitrary sequence $(\xi_j)_{j=1}^{\infty} \subset \mathcal{E}$ such that $\xi_j \to \delta_t$ in $\mathcal{E}'$ (notice that $\mathcal{E}$ is dense in $\mathcal{E}'$). Denote by $< x, \xi_j > \cdot$ the operator of multiplication by the function $< x, \xi >$. Evidently,

$$\ll < x, \xi_j > \cdot\phi, \psi \gg = \ll < x, \xi >, \overline{\phi}\psi \gg .$$

It remains only to note that $< x, \xi_j > \to x(t)$ in $(\mathcal{E}_{\mathrm{P}})^{-1}$ and $< x, \xi_j > \cdot\phi \to x(t)\cdot\phi$ in $(\mathcal{E}_{\mathrm{P}})^{-1}$ (see [6, 14]). □

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